[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1944 A

Unique Paper Code : 32355444

Name of the Paper : Elements of Analysis

Name of the Course : Mathematics : Generic Elective for Honours

Semester : IV

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any **two** parts from each question.

1. a) Define countable and uncountable sets. Show that the set $\mathbb{Q}$ of all rational numbers is denumerable. (7.5)

b) (i) Show that if $a, b \in \mathbb{R}$, $max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$ and $min\{a, b\} = \frac{1}{2}(a + b - |a - b|)$.

(ii) Determine the set A of all real numbers x such that $|x - 1| < |x|$. (3.5+4)

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c) Let S be a bounded set in $\mathbb{R}$ and let S_0 be a nonempty subset of S.

Show that $\inf S \leq \inf S_0 \leq \sup S_0 \leq \sup S$. (7.5)

d) Using Archimedean property, show that $\sup\left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\} = 1$. (7.5)

2.a) Define a convergent sequence. Using the definition, show that $\lim_{n\to\infty}\left(1 + \frac{(-1)^n}{n}\right) = 1$.

(7.5)

b) State and prove the squeeze theorem for sequences. Hence, show that $\lim_{n\to\infty}\frac{\sin n}{n} = 0$.

(7.5)

c) (i) State Bolzano Weierstrass Theorem for sequences. Give an example of an unbounded sequence that has exactly two limit points.

(ii) Define limit superior and limit inferior of a sequence. For $\langle x_n\rangle = \langle(-1)^n + \frac{2}{n}\rangle$,

show that $\lim_{n\to\infty}\sup(-x_n) = -\lim_{n\to\infty}\inf(x_n)$. (2+5.5)

d) Define Monotone Sequences. Let the sequence $\langle x_n\rangle$ be defined by $x_1 = 1$ and

$x_{n+1} = 2 - \frac{1}{x_n}$ for $n \in \mathbb{N}$. Show that the sequence $\langle x_n\rangle$ is monotone and bounded.

Find its limit. (7.5)

3.a) State the Integral test. Show using integral test that the series $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)}$ is convergent. (6.5)

b) Check whether the series are convergent or divergent:

i) $\sum_{n=1}^{\infty}\frac{1}{3^n+1}$.

ii) $\sum_{n=1}^{\infty}\frac{n^2-1}{n^4}$. (3+3.5)

c) State D'Alembert Ratio test. Construct examples of p-series where the ratio test fails.

(6.5)

d) State Absolute Convergence test and Conditional Convergence test. Show that the converse of absolute convergence test is not true. (6.5)

4. a) Test the series $\sum_{n=1}^{\infty} \frac{(n+1)^{n^2}}{(n)^{n^2}}$ for convergence. (6)

b) Determine whether the following series converge or diverge:

i) $\sum_{n=1}^{\infty} \frac{1}{2n-1}$.

ii) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$. (3+3)

c) Check the convergence of the series $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$. (6)

d) Examine the series $\sum_{n=1}^{\infty} \frac{3^n+4^n}{5^n}$ for convergence. Find its sum. (6)

5.a) Define Radius of convergence of a power series. Give examples of the series having radius of convergence 0 and ∞. (5)

b) Find interval of convergence for the power series $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$. (5)

c) Using Differentiation theorem for power series, show that

$$\sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}$$ (5)

d) Define cosine and sine functions as sums of power series. Show that for every $x, y \in \mathbb{R}$.

(i) $S(x-y) = S(x)\,C(y) - C(x)S(y)$

(ii) $S'(x) = C(x)$ and $S(2x) = 2\,S(x)C(x)$. (2.5+2.5)

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6.a) Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2}$. (5)

b) Using the power series of the function $f(x) = \frac{1}{(1-x)}$ valid for $|x| < 1$ show that

$\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$. (5)

c) State Integration theorem for power series For $|x| < 1$, find a power series representation for $\frac{1}{1+x^2}$. Show that $\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$. (5)

d) Write the power series expansion for $f(x) = e^x$ about $x_0 = 0$. (5)