[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 4902 **H**

Unique Paper Code : 2352572401

Name of the Paper : Abstract Algebra

Name of the Course : **B.A./B.Sc. (Programme) with Mathematics as Non-Major/ Minor – DSC**

Semester : IV

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Attempt **all** questions by selecting **two** parts from each question.

3. Each part carries **7.5** marks.

4. Use of Calculator not allowed.

1. (a) State Division Algorithm. Determine 51 mod 13, 342 mod 85, 62 mod 15, (82 . 73) mod 7, (51+68) mod 7 and (35 . 24) mod 11.

(b) Define a Group. Show that G = {1, –1, i, –i} forms a group under complex multiplication.

(c) Let G be a group with the property that for any x, y, z in the group, xy = zx implies y = z. Prove that G is Abelian. Also, in $GL(2, \mathbb{Z}_{13})$, find

$$\det\begin{bmatrix} 7 & 4 \\ 1 & 5 \end{bmatrix}.$$

2. (a) Let G be an abelian group and H and K be subgroups of G. Show that HK = {hk: h∈H, k∈K is a subgroup of G. Also, find the order of 7 in $\mathbb{Z}_{10}$ under addition modulo 10.

(b) Let a be an element in a group G of order 30. Find $\langle a^{26}\rangle$, $\langle a^{17}\rangle$, $\langle a^{18}\rangle$ and $|a^{26}|$, $|a^{17}|$ and $|a^{18}|$.

(c) Find the order of each element of U(15).

3. (a) Let $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 6 \end{bmatrix}$ and

$$\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{bmatrix}$$

Write α, β and αβ as a product of disjoint cycles. Also find β^{-1}.

(b) Construct a complete Cayley table for D_4, the group of symmetries of a square. Is D_4 Abelian? Justify.

(c) Find all the left cosets of $\{1,15\}$ in U(32).

4. (a) Let $H = \left\{\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, b, d \in \mathbb{R}, ad \neq 0\right\}$. Is H a normal subgroup of GL(2, $\mathbb{R}$)? Justify.

(b) State Lagrange's theorem for finite groups. Prove that in a finite group, the order of each element of the group divides the order of the group.

(c) Let G be a group of permutations. For each σ in G, define

$$sgn(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is an even permutation,} \\ -1 & \text{if } \sigma \text{ is an odd permutation.} \end{cases}$$

Prove that the function sgn is a homomorphism from G to the multiplicative group $\{-1,1\}$. What is the Kernel?

5. (a) (i) Describe all the subrings of the ring of integers.

(ii) Let a belong to a ring R. Let S = {x ∈ R | ax = 0). Show that S is a subring of R.

(b) Prove that a finite Integral Domain is a field. Hence, show that $\mathbb{Z}_p$ is a field, where p is a prime number.

(c) State and prove the subring test.

Let $R = \left\{ \begin{bmatrix} a & a-b \\ a-b & b \end{bmatrix} \mid a, b \in \mathbb{Z} \right\} \subseteq M_2(\mathbb{Z})$ where $M_2(\mathbb{Z})$ is the ring of 2 × 2 matrices over 2. Prove or disprove that R is a subring of $M_2(\mathbb{Z})$.

6. (a) Define an ideal of a ring R. State the ideal test. Hence, prove that $n\mathbb{Z}$ is an ideal of $\mathbb{Z}$.

(b) Let f : R → S be a ring homomorphism. Prove that

(i) f(A) is a subring of S where A is a subring of R.

(ii) $f^{-1}(B) = \{r \in R \mid f(r) \in B\}$ is an ideal of R where B is an ideal of S.

(c) Determine all the ring homomorphisms from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{30}$.